Design of Digital Circuits
Reading: Binary Numbers

Required Reading
for Week 1
20-21 February 2020
Spring 2020
Binary Numbers

Design of Digital Circuits 2016
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Frank K. Gürkaynak

http://www.syssec.ethz.ch/education/Digitaltechnik_16
In This Lecture

- How to express numbers using only 1s and 0s
- Using hexadecimal numbers to express binary numbers
- Different systems to express negative numbers
- Adding and subtracting with binary numbers
Number Systems

- Decimal Numbers
  
  $5374_{10} =$

- Binary Numbers
  
  $1101_2 =$
Number Systems

- **Decimal Numbers**

\[ 5374_{10} = 5 \times 10^3 + 3 \times 10^2 + 7 \times 10^1 + 4 \times 10^0 \]

<table>
<thead>
<tr>
<th>column</th>
<th>place value</th>
<th>digit</th>
<th>value</th>
</tr>
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<tbody>
<tr>
<td>1's</td>
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<tr>
<td>1000's</td>
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- **Binary Numbers**

\[ 1101_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 13_{10} \]

<table>
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<tr>
<th>column</th>
<th>place value</th>
<th>digit</th>
<th>value</th>
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# Powers of two

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Powers of two

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Handy to memorize up to $2^{15}$
Binary to Decimal Conversion

- Convert $10011_2$ to decimal
Binary to Decimal Conversion

- Convert $10011_2$ to decimal

$$2^4 \times 1 + 2^3 \times 0 + 2^2 \times 0 + 2^1 \times 1 + 2^0 \times 1 =$$
Binary to Decimal Conversion

- Convert $10011_2$ to decimal

\[
2^4 \times 1 + 2^3 \times 0 + 2^2 \times 0 + 2^1 \times 1 + 2^0 \times 1 = \\
16 \times 1 + 8 \times 0 + 4 \times 0 + 2 \times 1 + 1 \times 1 = \\
16 + 0 + 0 + 2 + 1 = 19_{10}
\]
Decimal to Binary Conversion

- Convert $47_{10}$ to binary
Decimal to Binary Conversion

Convert $47_{10}$ to binary

- Start with $2^6 = 64$ is $64 \leq 47$? no do nothing
- Now $2^5 = 32$
Decimal to Binary Conversion

Convert $47_{10}$ to binary

- Start with $2^6 = 64$ is $64 \leq 47$ ? no do nothing
- Now $2^5 = 32$ is $32 \leq 47$ ? yes subtract $47 - 32 = 15$
- Now $2^4 = 16$ is $16 \leq 15$ ? no do nothing
- Now $2^3 = 8$ is $8 \leq 15$ ? yes subtract $15 - 8 = 7$
- Now $2^2 = 4$ is $4 \leq 7$ ? yes subtract $7 - 4 = 3$
- Now $2^1 = 2$ is $2 \leq 3$ ? yes subtract $3 - 2 = 1$
- Now $2^0 = 1$ is $1 \leq 1$ ? yes we are done
Decimal to binary conversion

- **Convert $47_{10}$ to binary**
  - Start with $2^6 = 64$ is $64 \leq 47$ ? no 0 do nothing
  - Now $2^5 = 32$ is $32 \leq 47$ ? yes 1 subtract $47 - 32 = 15$
  - Now $2^4 = 16$ is $16 \leq 15$ ? no 0 do nothing
  - Now $2^3 = 8$ is $8 \leq 15$ ? yes 1 subtract $15 - 8 = 7$
  - Now $2^2 = 4$ is $4 \leq 7$ ? yes 1 subtract $7 - 4 = 3$
  - Now $2^1 = 2$ is $2 \leq 3$ ? yes 1 subtract $3 - 2 = 1$
  - Now $2^0 = 1$ is $1 \leq 1$ ? yes 1 we are done

- **Result is $0101111_2$**
Binary Values and Range

- **N-digit decimal number**
  - How many values? \(10^N\)
  - Range? \([0, 10^N - 1]\)
  - Example: 3-digit decimal number
    - \(10^3 = 1000\) possible values
    - Range: \([0, 999]\)

- **N-bit binary number**
  - How many values? \(2^N\)
  - Range? \([0, 2^N - 1]\)
  - Example: 3-digit binary number
    - \(2^3 = 8\) possible values
    - Range: \([0, 7] = [000_2\text{ to } 111_2]\)
### Hexadecimal (Base-16) Numbers

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<th>Hexadecimal</th>
<th>Binary</th>
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<td>C</td>
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<td>14</td>
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<tr>
<td>15</td>
<td>F</td>
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<tr>
<td>15</td>
<td>F</td>
<td>1111</td>
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</tbody>
</table>
Hexadecimal Numbers

- Binary numbers can be pretty long.

- A neat trick is to use base 16

- How many binary digits represent a hexadecimal digit?
  
  4 (since \(2^4 = 16\))

- Example 32 bit number:

  0101 1101 0111 0001 1001 1111 1010 0110
Hexadecimal Numbers

- Binary numbers can be pretty long.
- A neat trick is to use base 16
- How many binary digits represent a hexadecimal digit?
  - 4 (since $2^4 = 16$)
- Example 32 bit number:
  
<p>| | | | | | | | |</p>
<table>
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<td>0111</td>
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<td>1001</td>
<td>1111</td>
<td>1010</td>
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<tr>
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<td>D</td>
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<td>1</td>
<td>9</td>
<td>F</td>
<td>A</td>
<td>6</td>
</tr>
</tbody>
</table>
Hexadecimal Numbers

- Binary numbers can be pretty long.
- A neat trick is to use base 16
- How many binary digits represent a hexadecimal digit?
  4 (since $2^4 = 16$)

- Example 32 bit number:
  0101 1101 0111 0001 1001 1111 1010 0110
  5    D    7    1    9    F    A    6

- The other way is just as simple
  C    E    2    8    3    5    4    B
Hexadecimal Numbers

- Binary numbers can be pretty long.

- A neat trick is to use base 16.

- How many binary digits represent a hexadecimal digit?
  4 (since $2^4 = 16$)

Example 32 bit number:

```
0101 1101 0111 0001 1001 1111 1010 0110
5   D   7   1   9   F   A   6
```

The other way is just as simple:

```
C   E   2   8   3   5   4   B
1100 1110 0010 1000 0011 0101 0100 1011
```
Hexadecimal to Decimal Conversion

- Convert $4AF_{16}$ (or 0x4AF) to decimal
Hexadecimal to decimal conversion

- Convert $4AF_{16}$ (or 0x4AF) to decimal

\[
\begin{align*}
16^2 \times 4 + 16^1 \times A + 16^0 \times F &= \\
256 \times 4 + 16 \times 10 + 1 \times 15 &= \\
1024 + 160 + 15 &= 1199_{10}
\end{align*}
\]
Bits, Bytes, Nibbles...

10010110

most significant bit

least significant bit

byte

10010110
	nibble

CEBF9AD7

most significant byte

least significant byte
Powers of Two

- $2^{10} = 1 \text{ kilo} \approx 1000$ (1024)
- $2^{20} = 1 \text{ mega} \approx 1 \text{ million}$ (1,048,576)
- $2^{30} = 1 \text{ giga} \approx 1 \text{ billion}$ (1,073,741,824)
Powers of Two (SI Compatible)

- $2^{10} = 1 \text{ kibi} \approx 1000 \quad (1024)$

- $2^{20} = 1 \text{ mebi} \approx 1 \text{ million} \quad (1,048,576)$

- $2^{30} = 1 \text{ gibi} \approx 1 \text{ billion} \quad (1,073,741,824)$
Estimating Powers of Two

- What is the value of $2^{24}$?

- How many values can a 32-bit variable represent?
Estimating Powers of Two

■ What is the value of $2^{24}$?

$2^4 \times 2^{20} \approx 16$ million

■ How many values can a 32-bit variable represent?

$2^2 \times 2^{30} \approx 4$ billion
Addition

- **Decimal**

  \[
  \begin{array}{c}
  3734 \\
  + 5168 \\
  \hline
  8902
  \end{array}
  \]

  
  
  **Carries:** 11

- **Binary**

  \[
  \begin{array}{c}
  1011 \\
  + 0011 \\
  \hline
  1110
  \end{array}
  \]

  
  
  **Carries:** 11
Add the Following Numbers

1001
+ 0101
---
1011
+ 0110
Add the Following Numbers

\[
\begin{array}{c}
1 \\
1001 \\
+ 0101 \\
\hline
1110
\end{array}
\quad
\begin{array}{c}
111 \\
1011 \\
+ 0110 \\
\hline
10001
\end{array}
\]

OVERFLOW!
Overflow

- Digital systems operate on a fixed number of bits
- Addition overflows when the result is too big to fit in the available number of bits
- See previous example of 11 + 6
Overflow (Is It a Problem?)

- Possible faults
- Security issues

The $7 billion Ariane 5 rocket, launched on June 4, 1996, veered off course 40 seconds after launch, broke up, and exploded. The failure was caused when the computer controlling the rocket overflowed its 16-bit range and crashed.

The code had been extensively tested on the Ariane 4 rocket. However, the Ariane 5 had a faster engine that produced larger values for the control computer, leading to the overflow.
Binary Values and Range

- **N-digit decimal number**
  - How many values? \(10^N\)
  - Range? \([0, 10^N - 1]\)
  - Example: 3-digit decimal number
    - \(10^3 = 1000\) possible values
    - Range: \([0, 999]\)

- **N-bit binary number**
  - How many values? \(2^N\)
  - Range: \([0, 2^N - 1]\)
  - Example: 3-digit binary number
    - \(2^3 = 8\) possible values
    - Range: \([0, 7] = [000_2\text{ to } 111_2]\)
Signed Binary Numbers

- Sign/Magnitude Numbers
- One’s Complement Numbers
- Two’s Complement Numbers
Sign/Magnitude Numbers

- 1 sign bit, N-1 magnitude bits

- Sign bit is the most significant (left-most) bit
  - Positive number: sign bit = 0
  - Negative number: sign bit = 1

\[
A : \{a_{N-1}, a_{N-2}, \ldots a_2, a_1, a_0\} \\
A = (-1)^{a_{n-1}} \sum_{i=0}^{n-2} a_i 2^i
\]

- Example, 4-bit sign/mag representations of ±6:
  - +6 =
  - -6 =

- Range of an N-bit sign/magnitude number:
Sign/Magnitude Numbers

■ 1 sign bit, N-1 magnitude bits

■ Sign bit is the most significant (left-most) bit
  ▪ Positive number: sign bit = 0
  ▪ Negative number: sign bit = 1

Example, 4-bit sign/mag representations of ±6:

+6 = 0110
-6 = 1110

■ Range of an N-bit sign/magnitude number:

\[-(2^{N-1}-1), 2^{N-1}-1\]
Problems of Sign/Magnitude Numbers

- Addition doesn’t work, for example -6 + 6:

\[
\begin{align*}
\text{1110} \\
+ \text{0110} \\
\hline
\text{10100} \quad \text{wrong!}
\end{align*}
\]

- Two representations of 0 (± 0):

\[
\begin{align*}
\text{1000} \\
\text{0000}
\end{align*}
\]

- Introduces complexity in the processor design
  (Was still used by some early IBM computers)
One’s Complement

- A negative number is formed by reversing the bits of the positive number (MSB still indicates the sign of the integer):

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<th>$2^6$</th>
<th>$2^5$</th>
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<td>1</td>
<td>-0</td>
</tr>
</tbody>
</table>

...
One’s Complement

- The range of n-bit one’s complement numbers is:
  \([-2^{n-1}-1, 2^{n-1}-1]\]
  8 bits: [-127,127]

- Addition:
  Addition of signed numbers in one's complement is performed using binary addition with end-around carry. If there is a carry out of the most significant bit of the sum, this bit must be added to the least significant bit of the sum:

- Example: 17 + (-8) in 8-bit one’s complement

  \[
  \begin{array}{c}
  \hline
  & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \text{(17)} \\
  + & 1 & 1 & 1 & 1 & 0 & 1 & 1 & \text{(-8)} \\
  \hline
  1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  \end{array}
  \]

  1

  \[
  \begin{array}{c}
  \hline
  & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \text{= (9)} \\
  \end{array}
  \]
Two’s Complement Numbers

- Don’t have same problems as sign/magnitude numbers:
  - Addition works
  - Single representation for 0

- Has advantages over one’s complement:
  - Has a single zero representation
  - Eliminates the end-around carry operation required in one's complement addition
Two’s Complement Numbers

- A negative number is formed by reversing the bits of the positive number (MSB still indicates the sign of the integer) and adding 1:

<table>
<thead>
<tr>
<th>$2^7$</th>
<th>$2^6$</th>
<th>$2^5$</th>
<th>$2^4$</th>
<th>$2^3$</th>
<th>$2^2$</th>
<th>$2^1$</th>
<th>$2^0$</th>
<th>Two’s Complement</th>
<th>Unsigned</th>
</tr>
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<tbody>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>127</td>
<td>127</td>
</tr>
</tbody>
</table>
Two’s Complement Numbers

- A negative number is formed by reversing the bits of the positive number (MSB still indicates the sign of the integer) and adding 1:

<table>
<thead>
<tr>
<th>2^7</th>
<th>2^6</th>
<th>2^5</th>
<th>2^4</th>
<th>2^3</th>
<th>2^2</th>
<th>2^1</th>
<th>2^0</th>
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<th>Unsigned</th>
</tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>255</td>
</tr>
</tbody>
</table>
Two’s Complement Numbers

- Same as unsigned binary, but the most significant bit (msb) has value of $-2^{N-1}$
  \[ I = \sum_{i=0}^{i=n-2} b_i 2^i - b_{n-1} 2^{n-1} \]
  - Most positive 4-bit number:
  - Most negative 4-bit number:

- The most significant bit still indicates the sign
  (1 = negative, 0 = positive)

- Range of an N-bit two’s comp number:
Two’s Complement Numbers

- Same as unsigned binary, but the most significant bit (msb) has value of $-2^{N-1}$
  \[
  I = \sum_{i=0}^{i=n-2} b_i 2^i - b_{n-1} 2^{n-1}
  \]
  - Most positive 4-bit number: 0111
  - Most negative 4-bit number: 1000

- The most significant bit still indicates the sign
  (1 = negative, 0 = positive)

- Range of an N-bit two’s comp number:
  \([-2^{N-1}, 2^{N-1}-1]\) 8 bits: [-128,127]
“Taking the Two’s Complement”

- How to flip the sign of a two’s complement number:
  - Invert the bits
  - Add one

- Example: Flip the sign of $3_{10} = 0011_2$
“Taking the Two’s Complement”

- How to flip the sign of a two’s complement number:
  - Invert the bits
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- Example: Flip the sign of \( 3_{10} = 0011_2 \)
  - Invert the bits \( 1100_2 \)
“Taking the Two’s Complement”

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- Example: Flip the sign of $3_{10} = 0011_2$
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“Taking the Two’s Complement”

How to flip the sign of a two’s complement number:
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Example: Flip the sign of $3_{10} = 0011_2$
- Invert the bits $1100_2$
- Add one $1101_2$

Example: Flip the sign of $-8_{10} = 11000_2$
“Taking the Two’s Complement”

How to flip the sign of a two’s complement number:
- Invert the bits
- Add one

Example: Flip the sign of $3_{10} = 0011_2$
- Invert the bits $1100_2$
- Add one $1101_2$

Example: Flip the sign of $-8_{10} = 11000_2$
- Invert the bits $00111_2$
- Add one $01000_2$
Two’s Complement Addition

- Add 6 + (-6) using two’s complement numbers

```
  0110
+ 1010
```

- Add -2 + 3 using two’s complement numbers

```
1110
+ 0011
```
Two’s Complement Addition

- Add 6 + (-6) using two’s complement numbers
  \[
  \begin{array}{c}
  111 \\
  0110 \\
  + 1010 \\
  \end{array}
  \begin{array}{c}
  10000 \\
  \end{array}
  \]

- Add -2 + 3 using two’s complement numbers
  \[
  \begin{array}{c}
  111 \\
  1110 \\
  + 0011 \\
  \end{array}
  \begin{array}{c}
  10001 \\
  \end{array}
  \]

- Correct results if overflow bit is ignored
Increasing Bit Width

- A value can be extended from N bits to M bits (where M > N) by using:
  - Sign-extension
  - Zero-extension
Sign-Extension

- Sign bit is copied into most significant bits
- Number value remains the same
- Give correct result for two’s complement numbers

Example 1:
- 4-bit representation of 3 = 0011
- 8-bit sign-extended value: 00000011

Example 2:
- 4-bit representation of -5 = 1011
- 8-bit sign-extended value: 11111011
Zero-Extension

- Zeros are copied into most significant bits

- Value will change for negative numbers

**Example 1:**
- 4-bit value = $0011_2 = 3_{10}$
- 8-bit zero-extended value: $00000011_2 = 3_{10}$

**Example 2:**
- 4-bit value = $1011_2 = -5_{10}$
- 8-bit zero-extended value: $00001011_2 = 11_{10}$
# Number System Comparison

<table>
<thead>
<tr>
<th>Number System</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unsigned</td>
<td>[0, (2^N-1)]</td>
</tr>
<tr>
<td>Sign/Magnitude</td>
<td>([-2^{N-1}-1), 2^{N-1}-1]</td>
</tr>
<tr>
<td>Two’s Complement</td>
<td>([-2^{N-1}, 2^{N-1}-1]</td>
</tr>
</tbody>
</table>

For example, 4-bit representation:

<table>
<thead>
<tr>
<th>Decimal</th>
<th>Unsigned</th>
<th>Two’s Complement</th>
<th>Sign/Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>15</td>
<td>1111</td>
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</tr>
</tbody>
</table>
Lessons Learned

- How to express decimal numbers using only 1s and 0s
- How to simplify writing binary numbers in hexadecimal
- Adding binary numbers
- Methods to express negative numbers
  - Sign Magnitude
  - One’s complement
  - Two’s complement (the one commonly used)